

/ Proofs of Complex /

1] Prove that if the line joining the points z_1, z_2 and z_3, z_4 are perpendicular then:-

$$\operatorname{Re} \left(\frac{z_1 - z_2}{z_3 - z_4} \right) = 0$$

Sol

$$\arg(z_1 - z_2) = \theta_1$$

$$\arg(z_3 - z_4) = \theta_2$$

$$\theta_1 = \theta_2 + \frac{\pi}{2} \Rightarrow \theta_1 - \theta_2 = \frac{\pi}{2}$$

$$\arg(z_1 - z_2) - \arg(z_3 - z_4) = \frac{\pi}{2}$$

$$\arg \left(\frac{z_1 - z_2}{z_3 - z_4} \right) = \frac{\pi}{2}$$

یہ ہے نتیجہ $\frac{z_1 - z_2}{z_3 - z_4}$

$$\therefore \operatorname{Re} \left(\frac{z_1 - z_2}{z_3 - z_4} \right) = 0$$

2

Sub.
Date

2 use $z^n - 1 = 0$, $n = 2, 3, \dots$ to show that

a) $\cos \frac{2\pi}{n} + \cos \frac{4\pi}{n} + \dots + \cos \frac{2(n-1)\pi}{n} = -1$

b) $\sin \frac{2\pi}{n} + \sin \frac{4\pi}{n} + \dots + \sin \frac{2(n-1)\pi}{n} = 0$

$z^n - 1 = 0 \Rightarrow z = (1)^{\frac{1}{n}}$ Sol $\Rightarrow x=1, y=0, r=1, \theta=0$

$z = r e^{i(\frac{\theta + 2k\pi}{n})} = e^{\frac{2k\pi}{n}i}$

The roots $z_k = e^{\frac{2k\pi}{n}i}$, $k = 0, 1, \dots, n-1$

$(0 = z^{n-1})$ مجموع الجذور = 0

$z_0 + z_1 + z_2 + \dots + z_{n-1} = 0$

$e^0 + e^{\frac{2\pi i}{n}} + e^{\frac{4\pi i}{n}} + \dots + e^{\frac{2(n-1)\pi i}{n}} = 0$

$1 + \left[\cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right) \right] + \left[\cos\left(\frac{4\pi}{n}\right) + i \sin\left(\frac{4\pi}{n}\right) \right]$

$+ \dots + \left[\cos \frac{2(n-1)\pi}{n} + i \sin \frac{2(n-1)\pi}{n} \right] = 0$

نساوی الحقیقی بالحقیق :-

$$1 + \cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \dots + \cos\frac{2(n-1)\pi}{n} = 0$$

$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \dots + \cos\frac{2(n-1)\pi}{n} = -1 \quad \neq (a)$$

نساوی الخیالی بالتخیل :-

$$\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{4\pi}{n}\right) + \dots + \sin\frac{2(n-1)\pi}{n} = 0 \quad \neq (b)$$

[3] show that ~~(a)~~

(نساوی الحقیقی)

$$1 + \cos(\theta) + \cos(2\theta) + \dots + \cos(n\theta) = \frac{1}{2} + \frac{\sin\left(n + \frac{1}{2}\right)\theta}{2 \sin \frac{\theta}{2}}$$

Solution

$$e^{i\theta} = \cos \theta + i \sin \theta ; \operatorname{Re}[e^{i\theta}] = \cos \theta$$

$$\text{L.H.S. } 1 + \cos \theta + \cos 2\theta + \dots + \cos(n\theta)$$

$$= \operatorname{Re}\left[1 + e^{i\theta} + e^{i2\theta} + \dots + e^{in\theta}\right]$$

الماتريال الهندسية

$$a + ar + ar^2 + \dots = a \frac{1 - r^{n+1}}{1 - r}$$

here: $a = 1$, $r = e^{i\theta}$

$$\text{L.H.S} = \text{Re} \left[\frac{1 - e^{i\theta(n+1)}}{1 - e^{i\theta}} \right] \quad \frac{e^{-i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}} \quad \text{بالتقريب}$$

$$= \text{Re} \left[\frac{e^{-i\frac{\theta}{2}} - e^{i(n+1)\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}} \right]$$

$$= \text{Re} \left[\frac{\cos(\frac{\theta}{2}) - i \sin(\frac{\theta}{2}) - \cos[(n+1)\frac{\theta}{2}] - i \sin[(n+1)\frac{\theta}{2}]}{\cos(\frac{\theta}{2}) - i \sin(\frac{\theta}{2}) - \cos(\frac{\theta}{2}) - i \sin(\frac{\theta}{2})} \right]$$

$$= \frac{-\sin \frac{\theta}{2} - \sin(n+1) \frac{\theta}{2}}{-2 \sin \frac{\theta}{2}}$$

$$\text{L.H.S} = \frac{1}{2} + \frac{\sin(n+1)\theta}{2 \sin \frac{\theta}{2}} = \text{R.H.S} \quad \#$$

A] show that

مسألة

$$|Z_1 + Z_2|^2 + |Z_1 - Z_2|^2 = 2|Z_1|^2 + 2|Z_2|^2$$

Sol

$$\text{L.H.S.} = |Z_1 + Z_2|^2 + |Z_1 - Z_2|^2$$

$$= (Z_1 + Z_2)(\overline{Z_1 + Z_2}) + (Z_1 - Z_2)(\overline{Z_1 - Z_2})$$

$$= Z_1 \overline{Z_1} + Z_1 \overline{Z_2} + Z_2 \overline{Z_1} + Z_2 \overline{Z_2} + Z_1 \overline{Z_1} - Z_1 \overline{Z_2}$$

$$- Z_2 \overline{Z_1} + Z_2 \overline{Z_2}$$

$$\text{L.H.S.} = |Z_1|^2 + |Z_2|^2 + |Z_1|^2 + |Z_2|^2$$

$$\text{L.H.S.} = 2|Z_1|^2 + 2|Z_2|^2 = \text{R.H.S.}$$

///

[5] show that $f(z) = \bar{z} = x - iy$ is not differentiable at $z_0 = 0$

→ solution ←

$$f'(0) = \lim_{\Delta z \rightarrow 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z}, \quad f(z) = \bar{z}, \quad f(0) = 0$$

$$f(0 + \Delta z) = f(\Delta z) = \overline{\Delta z} = \Delta x + i\Delta y = \Delta x - i\Delta y$$

$$f'(0) = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$\lim_{\Delta x \rightarrow 0} \lim_{\Delta y \rightarrow 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} = 1$$

$$\lim_{\Delta y \rightarrow 0} \lim_{\Delta x \rightarrow 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{-i\Delta y}{i\Delta y} = -1$$

→ since $\lim \lim \neq \lim \lim$

∴ the limit doesn't exist

∴ not diff.

6 use C-R equations to show that

$$\frac{\partial z^n}{\partial z} = n z^{n-1}$$

Sol

مطلوب ان نوجد قانون المشتقة الأولى
(C-R) ونفرض اننا بعد الاختصار
تبقى $n z^{n-1}$

$$z = f(z) = z^n = (r e^{i\theta})^n = r^n e^{in\theta}$$

$$= r^n \cos(n\theta) + i r^n \sin(n\theta)$$

$u \leftarrow$ $\rightarrow v$

$$f'(z) = e^{-i\theta} \left[\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right]$$

$$= e^{-i\theta} \left[n r^{n-1} \cos(n\theta) + i n r^{n-1} \sin(n\theta) \right]$$

$$= n r^{n-1} e^{-i\theta} \left[\cos(n\theta) + i \sin(n\theta) \right]$$

$$e^{ix} = \cos x + i \sin x$$

$$f'(z) = n r^{n-1} e^{-i\theta} \left[e^{in\theta} \right]$$

$$\hat{f}(z) = n r^{n-1} e^{i(n-1)\theta} = n (r e^{i\theta})^{n-1}$$

$$\therefore \hat{f}(z) = n z^{n-1} \quad \neq$$

[7] Prove that $\frac{d}{dz} (z^2 \bar{z})$ doesn't exist any where.

[Sol]

$$f(z) = z^2 \bar{z} = (x+iy)^2 (x-iy)$$

$$= (x^2 + i2xy - y^2) (x-iy)$$

$$= x^3 + i2x^2y - xy^2 - i x^2y + 2xy^2 + i y^3$$

$$f(z) = \underbrace{(x^3 + x y^2)}_u + i \underbrace{(y^3 + x^2 y)}_v$$

$$u_x = 3x^2 + y^2$$

$$u_y = 2xy$$

$$v_x = 2xy$$

$$v_y = 3y^2 + x^2$$

$$u_x \neq v_y \quad (v_x \neq -u_y)$$

not analytic \Rightarrow not diff

Q. show that if $f(z) = u + iv$ is analytic

then:

$$\nabla^2 |f(z)|^2 = 4 \left| \frac{df}{dz} \right|^2$$

\downarrow ∇^2 $\xrightarrow{u^2+v^2}$ $\xrightarrow{u_x^2+v_x^2}$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Sol.

$$\frac{\partial}{\partial x} (u^2 + v^2) = 2u u_x + 2v v_x$$

$$\frac{\partial^2}{\partial x^2} (u^2 + v^2) = 2u u_{xx} + 2u_x^2 + 2v v_{xx} + 2v_x^2 \rightarrow (1)$$

$$\frac{\partial^2}{\partial y^2} (u^2 + v^2) = 2u u_{yy} + 2u_y^2 + 2v v_{yy} + 2v_y^2 \rightarrow (2)$$

(1) + (2)

$$= 2u (u_{xx} + u_{yy}) + 2(u_x^2 + v_x^2)$$

$$+ 2u (v_{xx} + v_{yy}) + 2(u_y^2 + v_y^2)$$

$$\therefore u_{xx} + u_{yy} = 0 \quad \text{و} \quad v_{xx} + v_{yy} = 0$$

$$u_x^2 + v_x^2 = F'(z)$$

$$u_y^2 + v_y^2 = F'(z)$$

$$\textcircled{1} + \textcircled{2} = 4 \left| \frac{dF}{dz} \right|^2$$

9

مسألة إثبات

a) show that if $F(z) = u(x, y) + i v(x, y)$ is analytic then $u(x, y)$ and $v(x, y)$ are harmonic

b) show that $F \rightarrow$ solution \leftarrow
* analytic function:-

$$u_x = v_y \rightarrow (1) \quad \text{و} \quad u_y = -v_x \rightarrow (2)$$

* harmonic function:-

$$u_{xx} + u_{yy} = 0$$

من معادلات (1) بالنسبة لـ x

$$u_{xx} = -v_{yy}$$

تفاضل (e) بالنسبة لـ y

$$u_{yy} = -v_{xy} \longrightarrow (b)$$

$$a + b$$

$$u_{xx} + u_{yy} = 0 \text{ is harmonic } \neq$$

[10] show that if $F(z) = u(r, \theta) + iv(r, \theta)$ is analytic then $u(r, \theta)$ and $v(r, \theta)$ are harmonic

solution

→ analytic function:-

$$u_r = \frac{1}{r} v_\theta \quad , \quad v_r = -\frac{1}{r} u_\theta$$

→ Harmonic Function.

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$$

$$\Rightarrow r u_r = v_\theta$$

تفاضل بالنسبة لـ r

$$r u_{rr} + u_r = v_{\theta r} \longrightarrow (1)$$

$$\Rightarrow r v_r = -u_\theta$$

تفاضل بالنسبة لـ \theta

$$r v_{\theta r} = -u_{\theta\theta} \longrightarrow (2)$$

$$V_{r\theta} = V_{\theta r} \text{ with (1), (2)}$$

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0 \quad \#$$

II] show that $\ln \frac{x+iy}{x-iy} = 2i \tan^{-1} \frac{y}{x}$

[sol]

$$Z = x+iy = r e^{i\theta} \quad (\bar{Z} = x-iy = r e^{-i\theta})$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1} \frac{y}{x}$$

$$\ln \frac{x+iy}{x-iy} = \ln \frac{r e^{i\theta}}{r e^{-i\theta}}$$

$$= \ln e^{2i\theta} = 2i\theta$$

$$= 2i \tan^{-1} \frac{y}{x}$$

#

Q12 show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$

Solution

$$\cos z = \cos(x+iy)$$

$$= \cos(x)\cos(iy) - \sin(x)\sin(iy)$$

$$= \cos(x) \cdot \cosh(y) - \sin(x) \sinh(y)$$

$$|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$

$$\because \sin^2 x = 1 - \cos^2 x \quad \& \quad \cosh^2 y = 1 + \sinh^2 y$$

$$|\cos z|^2 = \cos^2 x (1 + \sinh^2 y) + (1 - \cos^2 x) \sinh^2 y$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \quad \neq$$

[13] show that

$$\cosh^{-1} z = \ln \left(z \pm \sqrt{z^2 - 1} \right)$$

[Sol]

assume:

$$\cosh^{-1} z = w \Rightarrow z = \cosh w$$

$$z = \frac{e^w + e^{-w}}{2} \Rightarrow e^w + e^{-w} = 2z$$

$$(e^w)^2 - 2ze^w + 1 = 0$$

$$\text{Roots} = \frac{2z \pm \sqrt{4z^2 - 4}}{2} = e^w \pm 1$$

$$w = \ln \left(z \pm \sqrt{z^2 - 1} \right)$$

$$\therefore \cosh^{-1} z = \ln \left(z \pm \sqrt{z^2 - 1} \right)$$

14] show that:-

$$\frac{d}{dz} (\ln(z)) = \frac{1}{z}$$

[Sol]

$$f(z) = \ln z = \underbrace{\ln(r)}_{u} + \frac{i\theta}{\underbrace{1}_{v}}$$

$$u_r = \frac{1}{r} \quad \frac{1}{r} \rightarrow u_\theta = 0$$

$$v_r = 0 \quad \frac{1}{r} \rightarrow v_\theta = 1$$

f_n is analytic

$$\tilde{f}(z) = (u_r + i v_r) e^{-i\theta} = \frac{1}{r} e^{-i\theta}$$

$$= \frac{1}{r e^{i\theta}} \quad \text{where: } r e^{i\theta} = z$$

$$\tilde{f}(z) = \frac{1}{z}$$

#

[15] show that $(1+i)^i = e^{-\left(\frac{\pi}{4} \pm 2n\pi\right)} \cdot e^{\frac{i}{2} \ln(2)}$

solution

$$\text{L.H.S} = (1+i)^i = e^{\ln(1+i)^i} = e^{i \ln(1+i)}$$

$$= \cos(\ln(1+i)) + i \sin(\ln(1+i))$$

$$\ln(1+i) \quad r = \sqrt{2} \quad \theta = \frac{\pi}{4}$$

$$\ln(1+i) = \ln(\sqrt{2}) + i\left(\frac{\pi}{4} \pm 2n\pi\right)$$

$$\text{L.H.S} = e^{i\left[\ln\sqrt{2} + i\left(\frac{\pi}{4} \pm 2n\pi\right)\right]}$$

$$= e^{i \ln(\sqrt{2})} \cdot e^{-\left(\frac{\pi}{4} \pm 2n\pi\right)}$$

$$\sqrt{2} = (2)^{\frac{1}{2}}$$

$$= e^{\frac{i}{2} \ln(2)} \cdot e^{-\left(\frac{\pi}{4} \pm 2n\pi\right)}$$

$$= \text{R.H.S} \quad \#$$

16 Show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

Solution

$$\sin(z) = \sin(x+iy)$$

$$= \sin(x) \cos(iy) + \cos(x) \sin(iy)$$

$$= \sin(x) \cdot \cosh(y) + \cos(x) \cdot \sinh(y)$$

$$|\sin z|^2 = \sin^2 x \cdot \cosh^2 y + \cos^2 x \sinh^2 y$$

$$= \sin^2 x [1 + \sinh^2 y] + (1 - \sin^2 x) \sinh^2 y$$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \quad \text{//}$$

[17] show that

$$\sinh^{-1} z = \ln \left(z + \sqrt{z^2 + 1} \right)$$

L.H.S. $\sinh^{-1} z = w$

$$z = \sinh z = \frac{e^w - e^{-w}}{2} \Rightarrow e^w - e^{-w} = 2z$$

بالفرض $e^w = x$

$$e^{2w} - 2ze^w - 1 = 0$$

$$e^w = \frac{2z \pm \sqrt{4z^2 + 4}}{2} = z \pm \sqrt{z^2 + 1}$$

$$w = \ln \left(z \pm \sqrt{z^2 + 1} \right)$$

where $w = \sinh^{-1} z$

#

18] Prove that :-

$$\int_C (z - z_0)^m dz = \begin{cases} 2\pi i & m = -1 \\ 0 & \text{otherwise} \end{cases}$$

Where C is $|z - z_0| = a$

Sol

$$m \neq -1$$

$$|z - z_0| = a \Rightarrow z - z_0 = a e^{i\theta}$$

$$dz = i a e^{i\theta} d\theta \quad 0 \leq \theta \leq 2\pi$$

$$I = \int_0^{2\pi} (a e^{i\theta})^m i a e^{i\theta} d\theta$$

$$= i a^{m+1} \int_0^{2\pi} e^{i(m+1)\theta} d\theta$$

$$= i a^{m+1} \left[\frac{e^{i(m+1)\theta}}{i(m+1)} \right]_0^{2\pi}$$

$$I = \frac{ia^{m+1}}{i(m+1)} \left[\begin{matrix} e^{2\pi(m+1)i} \\ -1 \end{matrix} \right]$$

$$\therefore \sin(n\pi) = 0, \cos(n\pi) = (-1)^n$$

$$e^{2(m+1)\pi i} = \cos 2(m+1)\pi + i \sin 2(m+1)\pi$$

$$= (-1)^{2(m+1)} = 1$$

$$\therefore I = 0$$

at $m = -1$

$$I = \int_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{ia e^{i\theta} d\theta}{a e^{i\theta}} = 2\pi i$$

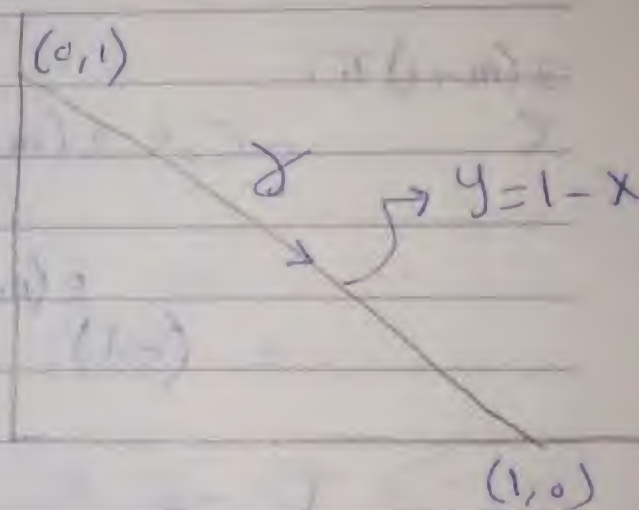
19 Let γ denote the line segment from $z=i$ to $z=1$.

Show that $\left| \int_{\gamma} \frac{dz}{z^4} \right| \leq 4\sqrt{2}$

Sol

$$\left| \int_{\gamma} \frac{dz}{z^4} \right| \leq ML$$

$$L(\gamma) = \sqrt{2}$$



$$|f(z)| = \left| \frac{1}{z^4} \right| = \frac{1}{|z|^4}$$

$$|z|^4 = \left(\sqrt{x^2 + y^2} \right)^4 = (x^2 + y^2)^2$$

$$= \left[x^2 + (1-x)^2 \right]^2 = (x^2 + 1 - 2x + x^2)^2$$

$$|z|^4 = 4 \left[x^2 - x + \frac{1}{2} \right]^2$$

الكمال مربع
(جذر الأول) الإشارة $\frac{1}{c}$ معامل الثاني $\left(\frac{1}{c}\right)$ معامل الثاني $+$ الباقي

$$|z|^4 = 4 \left[\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{1}{2} \right]^2$$

$$= 4 \left[\left(x - \frac{1}{2}\right)^2 + \frac{1}{4} \right]^2$$

$$|f(z)| = \frac{1}{4 \left[\left(x - \frac{1}{2}\right)^2 + \frac{1}{4} \right]^2} \quad 0 \leq x \leq 1$$

مع أعلى قيمة للمقدار عند $x = \frac{1}{2}$

$$|f(z)| \leq 4$$

$$\left| \int_{\gamma} f(z) dz \right| \leq ML$$

$$\leq 4\sqrt{2}$$

20) show that $\left| \int_{\gamma} (e^z - \bar{z}) dz \right| \leq 60$, where γ denote the boundary of triangle with vertices $z=0$, $z=-4$ and $z=3i$

solution

$$f(z), |e^z - \bar{z}| \leq |e^z| + |\bar{z}|$$

$$\leq |e^{x+iy}| + |x-iy|$$

$$\leq |e^x \cdot e^{iy}| + \sqrt{x^2 + y^2}$$

$$\leq e^x |\cos(y) + i \sin(y)| + \sqrt{x^2 + y^2}$$

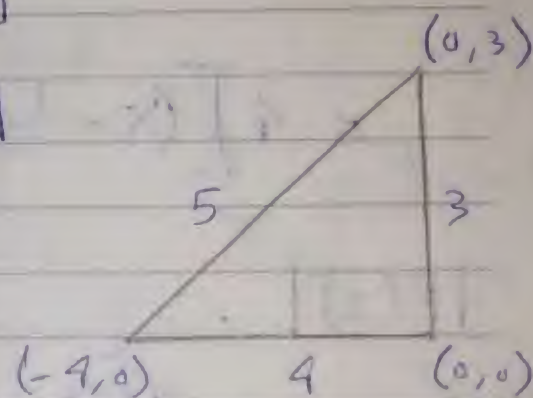
$$\leq e^x \sqrt{\cos^2 y + \sin^2 y} + \sqrt{x^2 + y^2}$$

$$\leq e^x + \sqrt{x^2 + y^2}$$

$$\text{at } (0,0) \rightarrow |e^z - \bar{z}| \leq 1$$

$$\text{at } (0,3) \rightarrow |e^z - \bar{z}| \leq 4$$

$$\text{at } (-4,0) \rightarrow |e^z - \bar{z}| \leq 4.02 \rightarrow M$$



$$\int_{\gamma} |f(z)| dz \leq 60 \quad \#$$

(21) show that if $f(z)$ is analytic on simple closed curve then $\oint_C f(z) dz = 0$

Note that

solution

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$\oint_C f(z) dz = \oint_C (u + iv)(dx + idy)$$

$$= \oint_C (u dx - v dy) + i \oint_C (v dx + u dy)$$

$$= \iint_D \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy$$

Since $f(z)$ is analytic $\rightarrow u_x = v_y, u_y = -v_x$

$$\oint_C f(z) dz = 0 \quad \#$$

22] show that if $f(z)$ is analytic on $S.C.C$ and z_0 inside C then

$$\oint \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

[Sol]

فرض دائرة مركزها z_0 ونسبها r

$$|z-z_0| = r$$

$$\oint = \oint_{C_1}$$

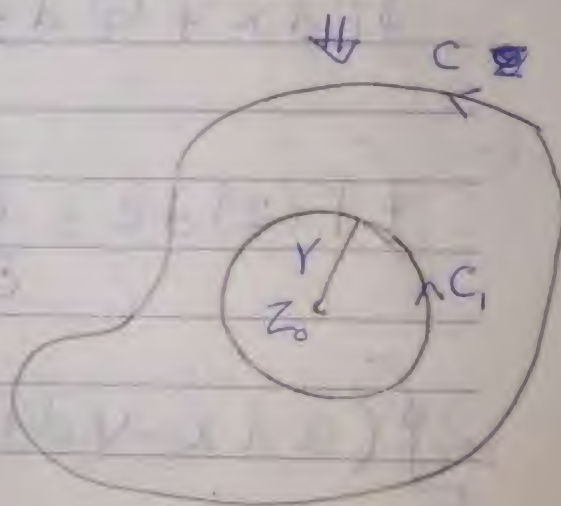
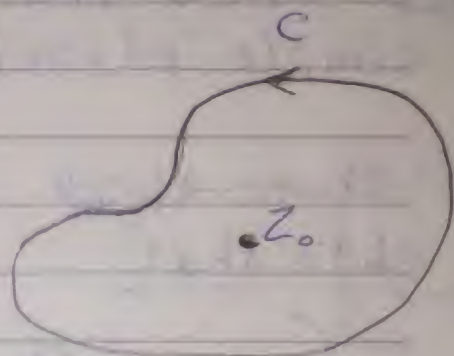
$$I = \oint \frac{f(z)}{z-z_0} dz$$

$$|z-z_0| = r$$

$$= \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta$$

as: $r \rightarrow 0$

$$I = i f(z_0) \int_0^{2\pi} d\theta = 2\pi i f(z_0) \quad \#$$



[23] Show that if $f(z)$ is analytic and bounded then $f(z)$ must be constant - use Cauchy integral form.

[Sol]

C. I

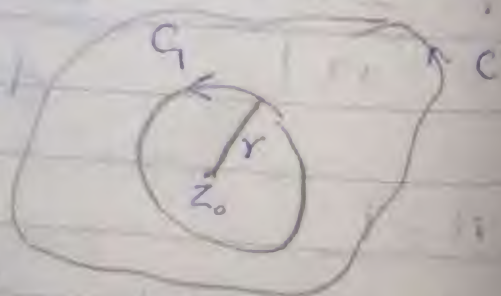
$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} \left. \frac{d^n f(z)}{dz^n} \right|_{z=z_0}$$

$$|f(z)| \leq M$$

فالدالة تكون محدودة إذا كان

معرفة دائرة مركزها z_0 ونصف قطرها (r) داخل C .

$$\oint = \oint_{C_1}$$



Note ~~if~~ $\left| \int f \right| \leq \int |f|$

$$\left| \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \right| \leq \oint \frac{|f(z)|}{|z-z_0|^{n+1}} dz$$

$$|z-z_0| = r$$

$$|f|_{c_1} \leq \frac{M}{r^{n+1}} \oint_C |dz|$$

وحدہ طول (جوز منہ الدائرہ وجميعها يعطى محيط الدائرہ بالکامل)

$$|f|_{c_1} \leq \frac{M}{r^{n+1}} (2\pi r)$$

$$\leq \frac{M}{r^n} (2\pi)$$

$$\frac{2\pi i}{n!} \left. \frac{d^n f(z)}{dz^n} \right|_{z=z_0} \leq \frac{2\pi M}{r^n}; |z|=r$$

$n=1$

$$|f'(z_0)| \leq \frac{M}{r}$$

as $r \rightarrow 0 \Rightarrow |f'(z_0)| \leq 0$ لا يوجد مقياس
سال

$$f'(z_0) = 0 \rightarrow f(z) = c \text{ for all } z.$$

$$f(z) = \text{constant} \neq$$

24 use Laurent's series to show that $\oint_C f(z) dz = 2\pi i a_{-1}$ where $f(z)$ is analytic on the region $r < |z - z_0| < R$

use

5.1

$$\oint_C (z - z_0)^m dz = \begin{cases} 2\pi i & m = -1 \\ 0 & m \neq -1 \end{cases} \rightarrow \textcircled{1}$$

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n$$

$$f(z) = \dots + a_{-1} (z - z_0)^{-1} + a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \dots$$

$(z - z_0)^m$ بالقرن \times

$$f(z) \times (z - z_0)^m = \dots + a_{-1} (z - z_0)^{m-1} + a_0 (z - z_0)^m + a_1 (z - z_0)^{m+1} + \dots$$

منه بالتالي كل على المنحنى واستخدام الدورة $\textcircled{1}$

$$\text{at } m=0 \Rightarrow \oint f(z) dz = a_{-1} (2\pi i)$$